Existences of rainbow matchings and rainbow matching covers.

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Abstract

Let G be an edge-coloured graph. A rainbow subgraph in G is a subgraph such that its edges have distinct colours. The minimum colour degree $\delta^c(G)$ of G is the smallest number of distinct colours on the edges incident with a vertex of G. We show that every edge-coloured graph G on $n \geq 7k/2+2$ vertices with $\delta^c(G) \geq k$ contains a rainbow matching of size at least k, which improves the previous result for $k \geq 10$.

Let $\Delta_{\text{mon}}(G)$ be the maximum number of edges of the same colour incident with a vertex of G. We also prove that if $t \geq 11$ and $\Delta_{\text{mon}}(G) \leq t$, then G can be edge-decomposed into at most $\lfloor tn/2 \rfloor$ rainbow matchings. This result is sharp and improves a result of LeSaulnier and West.

Keywords: edge coloring, rainbow, matching
2010 MSC: 05C15, 05C70

1. Introduction

Let G be a simple graph, that is, it has no loops or multi-edges. We write $V(G)$ for the vertex set of G and $\delta(G)$ for the minimum degree of G. An *edge-coloured graph* is a graph in which each edge is assigned a colour. We say that an edge-coloured graph G is *proper* if no two adjacent edges have the same colour. A subgraph H of G is *rainbow* if all its edges have distinct colours. Rainbow subgraphs are also called totally multicoloured, polychromatic, or heterochromatic subgraphs.

In this paper, we are interested in rainbow matchings in edge-coloured graphs. The study of rainbow matchings began with a conjecture of Ryser [10], which states that every Latin square of odd order contains a Latin transversal. Equivalently, for n odd, every properly n-edge-colouring of $K_{n,n}$, the complete

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[1]The research leading to these results was supported by the European Research Council under the ERC Grant Agreement no. 258345.

bipartite graph with n vertices on each part, contains a rainbow copy of a perfect matching. In a more general setting, given a graph H, we wish to know if an edge-coloured graph G contains a rainbow copy of H. A survey on rainbow matchings and other rainbow subgraphs in edge-coloured graphs can be found in [3].

For a vertex v of an edge-coloured graph G, the *colour degree*, $d^c(v)$, of v is the number of distinct colours on the edges incident with v. The smallest colour degree of all vertices in G is the *minimum colour degree of* G and is denoted by $\delta^c(G)$. Note that a properly edge-coloured graph G with $\delta(G) \geq k$ has $\delta^c(G) \geq k$.

Li and Wang [8] showed that if $\delta^c(G) = k$, then G contains a rainbow matching of size $\lceil (5k-3)/12 \rceil$. They further conjectured that if $k \geq 4$, then G contains a rainbow matching of size $\lceil k/2 \rceil$. LeSaulnier et al. [6] proved that if $\delta^c(G) = k$, then G contains a rainbow matching of size $\lfloor k/2 \rfloor$. The conjecture was later proved in full by Kostochka and Yancey [5].

Wang [11] asked does there exist a function $f(k)$ such that every properly edge-coloured graph G on $n \geq f(k)$ vertices with $\delta(G) \geq k$ contains a rainbow matching of size at least k. Diemunsch et al. [1] showed that such function does exist and $f(k) \leq 98k/23$. Gyárfás and Sarkozy [2] improved the result to $f(k) \leq 4k-3$. Independently, Tan and the author [9] showed that $f(k) \leq 4k-4$ for $k \geq 4$.

Kostochka, Pfender and Yancey [4] showed that every (not necessarily properly) edge-coloured G on $n \geq 17k^2/4$ vertices with $\delta^c(G) \geq k$ contains a rainbow matching of size k. Tan and the author [9] improved the bound to $n \geq 4k-4$ for $k \geq 4$. In this paper we show that $n \geq 7k/2 + 2$ is sufficient.

Theorem 1.1. *Every edge-coloured graph G on $n \geq 7k/2 + 2$ vertices with $\delta^c(G) \geq k$ contains a rainbow matching of size k.*

Moreover if G is bipartite, then we further improve the bound to $n \geq (3 + \varepsilon)k + \varepsilon^{-2}$.

Theorem 1.2. *Let $0 < \varepsilon \leq 1/2$ and $k \in \mathbb{N}$. Every edge-coloured bipartite graph G on $n \geq (3+\varepsilon)k + \varepsilon^{-2}$ vertices with $\delta^c(G) \geq k$ contains a rainbow matching of size k.*

We also consider covering an edge-coloured graph G by rainbow matchings. Given an edge-coloured graph G, let $\Delta_{\text{mon}}(G)$ be the largest maximum degree of monochromatic subgraphs of G. LeSaulnier and West [7] showed that every edge-coloured graph G on n vertices with $\Delta_{\text{mon}}(G) \leq t$ has an edge-decomposition into at most $t(1+t)n \ln n$ rainbow matchings. We show that G can be edge-decomposed into $\lfloor tn/2 \rfloor$ rainbow matchings provided $t \geq 11$.

Theorem 1.3. *For all $t \geq 11$, every edge-coloured graph G on n vertices with $\Delta_{\text{mon}}(G) \leq t$ can be edge-decomposed into $\lfloor tn/2 \rfloor$ rainbow matchings.*

Note that the bound is best possible by considering edge-coloured graphs, where one colour class induces a t-regular graph.

Theorems 1.1 and 1.2 are proved in Section 2. Theorem 1.3 is proved in Section 3.

2. Existence of rainbow matchings

We write $[k]$ for $\{1, 2, \ldots, k\}$. Let G be a graph with an edge-colouring c. We denote by $c(G)$ the set of colours in G. We write $|G|$ for $|V(G)|$. Given $W \subseteq V(G)$, $G[W]$ is the induced subgraph of G on W. All colour sets are assumed to be finite.

Before proving Theorems 1.1 and 1.2, we consider the following (weaker) question. Suppose that G is an edge-coloured graph and contains a rainbow matching M of size $k-1$. Under what colour degree and $|G|$ conditions can we 'extend' M into a matching of size k with at least $k-1$ colours? We formalise the question below.

Let $\mathcal{G}$ be a family of graphs closed under vertex/edge deletions. Define $\gamma(\mathcal{G})$ to be the smallest constant γ such that, whenever $k \in \mathbb{N}$, $G \in \mathcal{G}$ is a graph with $|G| \geq \gamma k$ and an edge-colouring c on G, the following holds. If for any rainbow matching M of size $k-1$ in G, we have $d^c(z) \geq k$ for all $z \in V(G) \setminus V(M)$, then G contains a rainbow matching M' of size $k-1$ and a disjoint edge. (Note that the colour of the disjoint edge may appear in M'.) Clearly, $\gamma(\mathcal{G}) \geq 2$ for any family $\mathcal{G}$ of graphs. It is easy to see that equality holds if $\mathcal{G}$ is the family of bipartite graphs.

Proposition 2.1. *Let $\mathcal{G}$ be the family of bipartite graphs. Then $\gamma(\mathcal{G}) = 2$.*

Proof. Let G be a bipartite graph on at least $2k$ vertices. Suppose that M is a rainbow matching of size $k-1$ and that $d^c(z) \geq k$ for all $z \in V(G) \setminus V(M)$. Since G is bipartite, there exists an edge vertex-disjoint from M and so the proposition follows. □

If $\mathcal{G}$ is the family of all graphs, we will show that $\gamma(\mathcal{G}) \leq 3$.

Lemma 2.2. *Let G be a graph with at least $3(k-1)+1$ vertices. Suppose that M is a rainbow matching of size $k-1$ and that $d^c(z) \geq k$ for all $z \in V(G) \setminus V(M)$. Then G contains a rainbow matching M' of size $k-1$ and a disjoint edge.*

Proof. Let $x_1y_1, \ldots, x_{k-1}y_{k-1}$ be the edges of M with $c(x_iy_i) = i$. Let $W = V(G) \setminus V(M)$. We may assume that $G[W]$ is empty or else the lemma holds easily.

Suppose the lemma does not hold for G. By relabeling the indices of i and swapping the roles of x_i and y_i if necessary, we will show that there exist distinct vertices $z_1, \ldots, z_{k-1}$ in W such that for each $1 \leq i \leq k-1$, the following holds:

(a$_i$) y_iz_i is an edge and $c(y_iz_i) \notin [i]$.

(b$_i$) Let T_i be the vertex set $\{x_j, y_j, z_j : i \leq j \leq k-1\}$. For any colour j', there exists a rainbow matching $M^i_{j'}$ of size $k-i$ on T_i such that $c(M^i_{j'}) \cap ([i-1] \cup \{j'\}) = \emptyset$.

(c$_i$) Let $W_i = W \setminus \{z_i, z_{i+1}, \ldots, z_{k-1}\}$. For all $w \in W_i$, $N(w) \cap T_i \subseteq \{y_i, \ldots, y_{k-1}\}$.

Let $W_k = W$ and $T_k = \emptyset$. Suppose that we have already found $z_{k-1}, z_{k-2}, \ldots, z_{i+1}$. We find z_i as follows.

Note that $|W_{i+1}| \geq n - 2(k-1) - (k-i-1) \geq 1$, so $W_{i+1} \neq \emptyset$. Let z be a vertex in W_{i+1}. By the colour degree condition, z must incident to at least k edges of distinct colours, and in particular, at least $k-i$ distinct coloured edges not using colours in $[i]$. By (c_{i+1}), z sends at most $k-i-1$ edges to T_{i+1}. So there exists a vertex $u \in V(M) \setminus T_{i+1} = \{x_j, y_j : 1 \leq j \leq i\}$ such that uz is an edge with $c(uz) \notin [i]$. Without loss of generality, $u = y_i$ and we set $z_i = z$. Clearly (a_i) holds.

We now show that (b_i) holds for any colour j'. If $j' \neq i$, then by (b_{i+1}), there is a rainbow matching $M_{j'}^{i+1}$ of size $k-i-1$ on T_{i+1} such that $c(M_{j'}^{i+1}) \cap ([i] \cup \{j'\}) = \emptyset$. Set $M_{j'}^{i} = M_{j'}^{i+1} \cup x_i y_i$. So $M_{j'}^{i}$ is a rainbow matching on T_i of size $k-i$ and moreover $c(M_{j'}^{i}) \cap ([i-1] \cup \{j'\}) = \emptyset$ as required. If $j' = i$, then by (b_{i+1}), there is a rainbow matching $M_{c(y_i z_i)}^{i+1}$ of size $k-i-1$ on T_{i+1} such that $c(M_{c(y_i z_i)}^{i+1}) \cap ([i] \cup \{c(y_i z_i)\}) = \emptyset$. Set $M_i^i = M_{c(y_i z_i)}^{i+1} \cup y_i z_i$. Note that M_i^i is the desired rainbow matching.

Let wt be an edge with $w \in W_i$ and $t \in T_i$. Since $G[W]$ is empty, $t \notin \{z_i, z_{i+1}, \ldots, z_{k-1}\}$. By ($c_{i+1}$), $t \notin \{x_{i+1}, x_{i+2}, \ldots, x_{k-1}\}$. Suppose that $t = x_i$. By (b_{i+1}), there exists a rainbow matching $M_{c(y_i z_i)}^{i+1}$ of size $k-i-1$ on T_{i+1} such that $c(M_{c(y_i z_i)}^{i+1}) \cap ([i] \cup \{c(y_i z_i)\}) = \emptyset$. Let M' be the matching $\{x_j y_j : j \in [i-1]\} \cup M_{c(y_i z_i)}^{i+1} \cup \{y_i z_i\}$. Note that M' is a rainbow matching of size $k-1$ vertex-disjoint from the edge wx_i. This contradicts the fact that G is a counterexample. Hence we have $t \in \{y_i, y_{i+1}, \ldots, y_{k-1}\}$ implying (c_i).

Therefore we have found $z_1, \ldots, z_{k-1}$. Let $w \in W_1 \neq \emptyset$. Recall the $G[W] = \emptyset$, so $N(w) \subseteq \{y_1, \ldots, y_{k-1}\}$ by (c_1), which implies that $d^c(w) \leq d(w) \leq k-1$, a contradiction. $\square$

Corollary 2.3. *Every family $\mathcal{G}$ of graphs satisfies $\gamma(\mathcal{G}) \leq 3$.*

For colour sets C and integers ℓ, we now define a (C, ℓ)-adapter below, which will be crucial in the proof of Lemma 2.5. Roughly speaking a (C, ℓ)-adapter is a vertex subset W that contains a rainbow matching M with $c(M) = C$ even after removing a vertex in W.

Given $\ell \in \mathbb{N}$ and a set C of colours, a vertex subset $W \subseteq V(G)$ is said to be a (C, ℓ)-*adapter* if there exist (not necessarily edge-disjoint) rainbow matchings $M_1, \ldots, M_\ell$ in $G[W]$ such that $c(M_i) = C$ for all $i \in [\ell]$, and given any $w \in W$, there exists $i \in [\ell]$ such that $w \notin V(M_i)$. We write C-adapter for $(C, |C|+1)$-adapter. Note that a (C, ℓ)-adapter is also a (C, ℓ')-adapter for all $\ell \leq \ell'$. The following proposition studies some basic properties of (C, ℓ)-adapters.

Proposition 2.4. *Let G be a graph with an edge-colouring c.*

(i) *Let $C = \{c_1, \ldots, c_\ell\}$ be a set of distinct colours. Let $W = \{x_i, y_i, z_i, w : i \in [\ell]\}$ be a vertex set such that $c(x_i y_i) = c_i = c(z_i w)$ for all $i \in [\ell]$. Then W is a C-adapter.*

(ii) *Let $\ell_1, \ldots, \ell_p \in \mathbb{N}$ and let $C_1, \ldots, C_p$ be pairwise disjoint colour sets. Suppose that W_j is a (C_j, ℓ_j)-adapter for all $j \in [p]$ and that $W_1, \ldots, W_p$ are pairwise disjoint. Then $\bigcup_{j=1}^p W_j$ is a $(\bigcup_{j=1}^p C_j, \max_{j \in [p]}\{\ell_j\})$-adapter.*

(iii) *Let C be a colour set. Suppose that W is a (C,ℓ)-adapter. Suppose that $x,y,z \in V(G)\setminus W$ and $w \in W$ such that $xy, zw \in E(G)$ and $c(xy) = c(zw) \notin C$. Then $W \cup \{x,y,z\}$ is a $(C \cup \{c(xy)\}, \ell+1)$-adapter.*

Proof. To prove (i), we simply set $M_i = \{x_j y_j : j \in [\ell] \setminus \{i\}\} \cup \{wz_i\}$ for all $i \in [\ell]$ and $M_{\ell+1} = \{x_j y_j : j \in [\ell]\}$.

(ii) Let $\ell = \max\{\ell_j : j \in [p]\}$. Note that each W_j is a (C_j, ℓ)-adapter. For $j \in [p]$, let $M_1^j, \dots, M_\ell^j$ be rainbow matchings in $G[W_j]$ such that $c(M_i^j) = C_j$ for all $i \in [\ell]$, and given any $w \in W_j$, there exists $i \in [\ell]$ such that $w \notin V(M_i^j)$. Set $M_i = \bigcup_{j=1}^p M_i^j$. So (ii) holds.

(iii) Let $M_1, \dots, M_\ell$ be rainbow matchings in $G[W]$ such that $c(M_i) = C$ for all $i \in [\ell]$, and given any $w' \in W$, there exists $i \in [\ell]$ such that $w' \notin V(M_i)$. Without loss of generality we have $w \notin V(M_1)$. Now set $M'_i = M_i \cup \{xy\}$ for all $i \in [\ell]$ and $M'_{\ell+1} = M'_1 \cup \{wz\}$. Hence, $W \cup \{x,y,z\}$ is a $(C \cup \{c(xy)\}, \ell+1)$-adapter. □

We prove the following lemma. The main idea of the proof is to consider (C,ℓ)-adapters in G with ℓ maximal.

Lemma 2.5. *Let $k \in \mathbb{N}$ and let $2 < \gamma \leq 3$. Let $\mathcal{G}$ be a family of graphs closed under vertex/edge deletion with $\gamma(\mathcal{G}) \leq \gamma$. Suppose that $G \in \mathcal{G}$ with*

$$|G| \geq \left(2 + \frac{\gamma}{2}\right) k + \frac{2(4-\gamma)}{(\gamma-2)^2} - 3 + \gamma$$

and that G contains a rainbow matching of size $k-1$. Further suppose that for all rainbow matchings M of size $k-1$ in G, we have $d^c(v) \geq k$ for all $v \in V(G) \setminus V(M)$. Then G contains a rainbow matching of size k.

Proof. We proceed by induction on k. It is trivial for $k=1$, so we may assume that $k \geq 2$.

Let $p \in \mathbb{N} \cup \{0\}$ and let $\ell_1, \dots, \ell_p \in \mathbb{N}$ with $\ell_1 \geq \dots \geq \ell_p$ and $\sum_{i=1}^p \ell_i \leq k-1$. Let $\mathcal{P} = \{W_1, \dots, W_p, U\}$ be a vertex partition of $V(G)$. We say that $\mathcal{P}$ has parameters $(\ell_1, \ell_2, \dots, \ell_p)$ if

(a) there exist p pairwise disjoint colour sets $C_1, \dots, C_p$ such that $|C_i| = \ell_i$ for all $i \in [p]$;
(b) W_i is a C_i-adapter and $|W_i| = 3\ell_i + 1$ for all $i \in [p]$;
(c) there exists a rainbow matching M_U of size $k - 1 - \sum_{i=1}^p \ell_i$ in $G[U]$ with $c(M_U) \cap C_i = \emptyset$ for all $i \in [p]$;
(d) $U \setminus V(M_U) \neq \emptyset$.

Since G contains a rainbow matching M of size $k-1$, such a vertex partition exists ($p = 0$ and $U = V(G)$ say). We now assume that $\mathcal{P}$ is chosen such that the string $(\ell_1, \dots, \ell_p)$ is lexicographically maximal. (Here, we view $(a_1, a_2, \dots, a_p)$ as $(a_1, a_2, \dots, a_p, 0, \dots, 0)$, e.g. $(3,2,2) \leq (4,1) \leq (4,1,1)$.)

Let $C_1, \dots, C_p$ be the sets of colours guaranteed by (a)–(c). Set $W = W_1 \cup \dots \cup W_p$ and $C = \bigcup_{i=1}^p C_i$. Let $\ell_0 = k - 1 - \sum_{i=1}^p \ell_i$. By (b) and

Proposition 2.4(ii), W is a (C, ℓ_1+1)-adapter. The following claim gives some useful properties of the rainbow matchings in $G[U]$ and $G \setminus W$. This will be needed to finish the proof of the lemma.

Claim 2.6. (i) *Let M_U be a rainbow matching of size ℓ_0 in $G[U]$ with $c(M_U) \cap C = \emptyset$. If $|U| \geq 2\ell_0 + 2$ and there is an edge $wz \in E(G)$ with $w \in W$ and $z \in U \setminus V(M_U)$, then we have $c(wz) \in C$.*

(ii) *Let M' be a rainbow matching of size $k-1-\ell_1$ in $G \setminus W$ with $c(M') \cap C_1 = \emptyset$. If $wx \in E(G)$ with $w \in W_1$ and $x \in V(G) \setminus (W_1 \cup V(M'))$, then $c(wx) \in C_1$.*

Proof of Claim. Suppose that (i) is false. There exists an edge $wz \in E(G)$ such that $c(wz) \notin C$, $w \in W_i$ for some $i \in [p]$ and $z \in U \setminus V(M_U)$. Note that there exists a rainbow matching M_W in $G[W \setminus w]$ such that $c(M_W) = C$ since W is a C-adapter. If $c(wz) \notin C \cup c(M_U)$, then $M_U \cup M_W \cup \{wz\}$ is a rainbow matching of size k, so we are done. If $c(wz) \in c(M_U)$, then let xy be the edge in M_U such that $c(xy) = c(wz)$. Set $W'_i = W_i \cup \{x, y, z\}$, $W'_j = W_j$ for all $j \in [p] \setminus \{i\}$ and $U' = U \setminus \{x, y, z\}$. Let $\ell'_i = \ell_i + 1$ and let $\ell'_j = \ell_j$ for all $j \in [p] \setminus \{i\}$. Set $C'_i = C_i \cup \{c(xy)\}$ and $C'_j = C_j$ for all $j \in [p] \setminus \{i\}$. By Proposition 2.4(iii), W'_j is a C'_j-adapter for all $j \in [p]$. Note that $M_{U'} = M_U - xy$ is a rainbow matching in $G[U']$ with $c(M_{U'}) \cap C'_j = \emptyset$ for all $j \in [p]$. Also $U' \setminus V(M_{U'}) = U \setminus (V(M_U) \cup \{z\}) \neq \emptyset$. By relabelling the sets W'_j and C'_j if necessary, we deduce that the vertex partition $\mathcal{P}' = \{W'_1, \dots, W'_p, U'\}$ has parameters $(\ell'_1, \dots, \ell'_p) > (\ell_1, \dots, \ell_p)$, which contradicts the maximality of $\mathcal{P}$. Hence (i) holds.

A similar argument proves (ii). $\square$

Suppose that $|U| > \gamma(\ell_0 + 1)$, so $|U| \geq 2\ell_0 + 3$. Let H be the resulting subgraph of $G[U]$ obtained after removing all edges of colours in C. Let M_U be a rainbow matching in H of size ℓ_0 with $c(M_U) \cap C = \emptyset$, which exists by (c). By Claim 2.6(i), we have for all $z \in V(H) \setminus V(M_U)$, $d^c_H(z) \geq k - |C| = \ell_0 + 1$. Since $\gamma(\mathcal{G}) \leq \gamma$, H contains a rainbow matching M_0 of size ℓ_0 and a disjoint edge e. If $c(e) = c(xy)$ for some $xy \in M_0$, then set $W_{p+1} = V(e) \cup \{x, y\}$, $C_{p+1} = \{c(xy)\}$, and $U' = U \setminus (V(e) \cup \{x, y\})$. Observe that W_{p+1} is a C_{p+1}-adapter by Proposition 2.4(i). Note that $M_0 - xy$ is a rainbow matching of size $\ell_0 - 1$ in $G[U']$ with $c(M_0) \cap \bigcup_{j \in [p+1]} C_j = \emptyset$ and $|U' \setminus V(M_0)| = |U| - 2\ell_0 - 2 \geq 1$. Hence the vertex partition $\mathcal{P}' = \{W_1, \dots, W_{p+1}, U'\}$ has parameters $(\ell_1, \dots, \ell_p, 1)$, contradicting the maximality of $\mathcal{P}$. If $c(e) \notin c(M_0)$, then $M_0 \cup e$ is a rainbow matching with $c(M_0 \cup e) \cap C = \emptyset$. Together with (b), G contains a rainbow matching of size k with colours $c(M_0 \cup e) \cup C$, so we are done. Therefore we may assume that

$$|U| \leq \gamma(\ell_0 + 1). \tag{1}$$

Since $2 < \gamma \leq 3$ and $\ell_0 \leq k - 1$, by the assumptions of Lemma 2.5, we have $|G| > (2 + \gamma/2)k > \gamma k \geq |U|$. Therefore, $W \neq \emptyset$ and $\ell_1 \geq 1$.

Next, suppose that $(\gamma-2)\ell_1 \geq 2$, so $|W_1| = 3\ell_1 + 1 \leq (2+\gamma/2)\ell_1$. Let H_1 be the subgraph of G obtained by removing all vertices of W_1 and all edges of colours in C_1. By the assumptions of Lemma 2.5, we then have

$$|H_1| = |G| - |W_1| \geq \left(2+\frac{\gamma}{2}\right)(k-\ell_1) + \frac{2(4-\gamma)}{(\gamma-2)^2} - 3 + \gamma.$$

By (b) and (c), H_1 contains a rainbow matching M' of size $k-1-\ell_1$. By Claim 2.6(ii), $c(wx) \in C_1$ for all $w \in W_1$ and $x \in V(H_1) \setminus V(M')$. Hence, $d^c_{H_1}(z) \geq k - |C_1| = k - \ell_1$ for all $z \in V(H_1) \setminus V(M')$. Note that this statement also holds for any rainbow matchings M' of size $k-1-\ell_1$ in H_1. Hence H_1 satisfies the hypothesis of the lemma with $k = k - \ell_1$. By the induction hypothesis, H_1 contains a rainbow matching M'' of size $k-\ell_1$. By (b), there exists a rainbow matching M_1 of size ℓ_1 in $G[W_1]$ such that $c(M_1) = C_1$. Since $c(M_1) \cap c(M'') \subseteq C_1 \cap c(H_1) = \emptyset$, $M_1 \cup M''$ is a rainbow matching of size k as required. Therefore we may assume that

$$(\gamma-2)\ell_1 < 2. \tag{2}$$

Recall that W is a (C, ℓ_1+1)-adapter. So there exist rainbow matchings M_1^*, M_2^*, ..., $M_{\ell_1+1}^*$ such that $c(M_i^*) = C$ for all $i \in [\ell_1+1]$ and

$$W = \bigcup_{i=1}^{\ell_1+1} (W \setminus V(M_i^*)). \tag{3}$$

Let M_U be a rainbow matching of size ℓ_0 in $G[U]$ with $c(M_U) \cap C = \emptyset$ (which exists by (c)). By (d), there exists $z \in U \setminus V(M_U)$. Note that z sends at least $d^c(z) - |V(M_U)| \geq k - 2\ell_0$ edges of distinct colours to $V(G) \setminus V(M_U)$. Let $q = \lceil (k-2\ell_0)/(\ell_1+1) \rceil$. By (3) and an averaging argument, there exists $i \in [\ell_1+1]$ such that there exist vertices $x_1, \dots, x_q \in V(G) \setminus V(M_U \cup M_i^*)$ such that $c(zx_j)$ is distinct for each $j \in [q]$. By Claim 2.6(i), we have $c(zx_j) \in C = c(M_i^*)$ for all $j \in [q]$. Let $e_1, \dots, e_q$ be edges of M_i^* such that $c(e_j) = c(zx_j)$ for all $j \in [q]$. Set $W' = \bigcup_{j\in[q]} (V(e_j) \cup \{x_j, z\})$ and $C' = \{c(e_j) : j \in [q]\}$. By Proposition 2.4(i), W' is a C'-adapter. Set $U' = V(G) \setminus W'$ and $M_{U'} = (M_i^* \cup M_U) \setminus W'$. Note that $V(M_{U'}) \subseteq U'$ and $M_{U'}$ is a rainbow matching of size $k-1-q$ with $c(M_{U'}) \cap C' = \emptyset$. Therefore, the vertex partition $\mathcal{P}' = \{W', U'\}$ has parameter (q). By the maximality of $\mathcal{P}$, we have $\ell_1 \geq q \geq (k-2\ell_0)/(\ell_1+1)$ and so

$$\ell_0 \geq (k - \ell_1(\ell_1+1))/2. \tag{4}$$

Recall that $|W_i| = 3\ell_i + 1 \leq 4\ell_i$ for all $i \in [p]$, that $\sum_{i=1}^{p} \ell_i + \ell_0 = k-1$, and

that $2 < \gamma \leq 3$. Finally, we have

$$\begin{aligned}|G| &= |W_1| + \sum_{i=2}^{p} |W_i| + |U| \overset{(1)}{\leq} 3\ell_1 + 1 + 4\sum_{i=2}^{p} \ell_i + \gamma(\ell_0 + 1)\\ &= 3\ell_1 + 1 + 4(k-1-\ell_1) - (4-\gamma)\ell_0 + \gamma\\ &\overset{(4)}{\leq} 4k - 3 - \ell_1 - \frac{(4-\gamma)(k-\ell_1(\ell_1+1))}{2} + \gamma\\ &= \left(2 + \frac{\gamma}{2}\right)k - 3 - \ell_1 + \frac{(4-\gamma)\ell_1(\ell_1+1)}{2} + \gamma\\ &< \left(2 + \frac{\gamma}{2}\right)k + \frac{(4-\gamma)\ell_1^2}{2} - 3 + \gamma \overset{(2)}{<} \left(2 + \frac{\gamma}{2}\right)k + \frac{2(4-\gamma)}{(\gamma-2)^2} - 3 + \gamma,\end{aligned}$$

a contradiction. This completes the proof of the lemma. $\square$

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. We first prove Theorem 1.1 by induction on k. Let G be an edge-coloured graph on $n \geq 7k/2 + 2$ vertices with $\delta^c(G) \geq k$. This is trivial for $k = 1$ and so we may assume that $k \geq 2$. By the induction hypothesis G contains a rainbow matching of size $k - 1$. Since $\delta^c(G) \geq k$, Corollary 2.3 implies that G satisfies the hypothesis of Lemma 2.5 with $\gamma = 3$. Therefore, G contains a rainbow matching of size k as required.

To prove Theorem 1.2, first note that by Proposition 2.1, $\gamma(\mathcal{G}') = 2$, where $\mathcal{G}'$ is the family of all bipartite graphs. Also, for $\gamma = 2 + 2\varepsilon$, we have

$$\left(2 + \frac{\gamma}{2}\right)k + \frac{2(4-\gamma)}{(\gamma-2)^2} - 3 + \gamma = (3+\varepsilon)k + \frac{2(2-2\varepsilon)}{4\varepsilon^2} - 1 + 2\varepsilon \leq (3+\varepsilon)k + \varepsilon^{-2}.$$

Therefore, Theorem 1.2 follows from a similar argument used in the preceding paragraph, where we take $\gamma = 2 + 2\varepsilon$ and $\mathcal{G}$ to be the family of all bipartite graphs in the application of Lemma 2.5. $\square$

We would like to point out that an improvement of Corollary 2.3 would lead to an improvement of Theorem 1.1. However, we believe that new ideas are needed to prove the case when $2k < |G| < 3k$.

3. Existence of rainbow matching covers

Proof of Theorem 1.3. By colouring every missing edge in G with a new colour, we may assume that G is an edge-coloured complete graph on n vertices with $\Delta_{\text{mon}}(G) = t$ and colours $\{1, 2, \ldots, p\}$. For $i \leq p$, let G^i be the subgraph of G induced by the edges of colour i. Without loss of generality, we may assume that $e(G^1) \geq e(G^2) \geq \cdots \geq e(G^p)$.

For $1 \leq i \leq p$, suppose that we have already found a set $\mathcal{M} = \{M_1, \ldots, M_{\lfloor tn/2 \rfloor}\}$ of edge-disjoint (possibly empty) rainbow matchings such that $\bigcup_{1 \leq j \leq \lfloor tn/2 \rfloor} M_j =$

$\bigcup_{j'<i} E(G^{j'})$. We now assign edges of G^i to these matchings so that the resulting rainbow matchings $M'_1, \dots, M'_{\lfloor tn/2 \rfloor}$ contain all edges of $G^1 \cup \cdots \cup G^i$. Define an auxiliary bipartite graph H as follows. The vertex classes of H are $E(G^i)$ and $\mathcal{M}$. An edge $f \in E(G^i)$ is joined to a rainbow matching $M_j \in \mathcal{M}$ if and only if f is vertex-disjoint from M_j. If H contains a matching of size $e(G^i)$, then we assign $f \in E(G^i)$ to $M_j \in \mathcal{M}$ according to the matching in H. Thus we have obtained the desired rainbow matchings $M'_1, \dots, M'_{\lfloor tn/2 \rfloor}$. Therefore, to prove the theorem, it is sufficient to show that H satisfies Hall's conditions.

Let $f \in E(G^i)$. Since f is incident to $2(n-2)$ edges in G, f is incident to at most $2(n-2)$ matchings $M_j \in \mathcal{M}$. Thus,

$$|N_H(f)| \geq |\mathcal{M}| - 2(n-2) \geq (t-4)n/2. \tag{5}$$

We divide the proof into two cases depending on the value of i.

Case 1: $i \leq \frac{(t-4)n}{4(t+1)}$**.** Let $S \subseteq E(G^i)$ with $S \neq \emptyset$. Note that each $M_j \in \mathcal{M}$ has size at most $i-1$. If S contains a matching of size $2i-1$, then for every $M_j \in \mathcal{M}$, there exists an edge $f \in S$ vertex-disjoint from M_j. Thus, $N_H(S) = \mathcal{M}$ and so $|N_H(S)| = \lfloor tn/2 \rfloor \geq e(G^i) \geq |S|$.

Therefore, we may assume that S does not contain a matching of size $2i-1$. By Vizing's theorem, $|S| \leq 2(i-1)(\Delta(G^i)+1) \leq 2(i-1)(t+1)$. By (5) and the assumption on i, we have

$$|N_H(S)| \geq (t-4)n/2 \geq 2(i-1)(t+1) \geq |S|.$$

Therefore, Hall's condition holds for this case.

Case 2: $i > \frac{(t-4)n}{4(t+1)}$**.** Since $e(G^1) \geq e(G^2) \geq \cdots \geq e(G^p)$, we have $e(G^i) \leq \binom{n}{2}/i < 2(t+1)n/(t-4)$. Let $S \subseteq E(G^i)$ with $S \neq \emptyset$. By (5) and the fact that $t \geq 11$, we have

$$|N_H(S)| \geq (t-4)n/2 \geq 2(t+1)n/(t-4) > e(G^i) \geq |S|.$$

Therefore, Hall's condition also holds for this case. This completes the proof of the theorem. □

Acknowledgements

The author would like to thank the referees for their helpful suggestions.

References

[1] J. Diemunsch, M. Ferrara, A. Lo, C. Moffatt, F. Pfender, and P. S. Wenger, *Rainbow matching of size $\delta(G)$ in properly-colored graphs*, Electronic Journal of Combinatorics **19** (2012), #P52.

[2] A. Gyárfás and G. N. Sarkozy, *Rainbow matchings and cycle-free partial transversals of Latin squares*, Discrete Mathematics **327** (2014), 96–102.

[3] M. Kano and X. Li, *Monochromatic and heterochromatic subgraphs in edge-colored graphs – a survey*, Graphs Combin. **24** (2008), 237–263.

[4] A. Kostochka and M. Yancey, *Large rainbow matchings in edge-coloured graphs*, Combinatorics, Probability and Computing **21** (2012), 255–263.

[5] A. Kostochka , F. Pfender and M. Yancey, *Large rainbow matchings in large graphs*, arXiv:1204.3193.

[6] T. D. LeSaulnier, C. Stocker, P. S. Wenger, and D. B. West, *Rainbow matching in edge-colored graphs*, Electronic Journal of Combinatorics **17** (2010), #N26.

[7] T. D. LeSaulnier and D. B. West, *Rainbow edge-coloring and rainbow domination*, Discrete Mathematics **313** (2013), 2020–2025.

[8] H. Li and G. Wang, *Heterochromatic matchings in edge-colored graphs*, Electronic Journal of Combinatorics **15** (2008), #R138.

[9] A. Lo and T. S. Tan, *A note on large rainbow matchings in edge-coloured graphs*, Graphs and Combinatorics **30** (2014), 389–393.

[10] H. J. Ryser, *Neuere probleme der kombinatorik*, Vortrage über Kombinatorik Oberwolfach, Mathematisches Forschungsinstitut Oberwolfach (1967), 24–29.

[11] G. Wang, *Rainbow matchings in properly edge colored graphs*, Electronic Journal of Combinatorics **18** (2011), #P162.